4.6 The Fundamental Theorem of Algebra

COMMON CORE

Learning Standards
HSN-CN.C.8
HSA-CN.C.9
HSA-APR.B.3

Essential Question

How can you determine whether a polynomial equation has imaginary solutions?

EXPLORATION 1 Cubic Equations and Imaginary Solutions

Work with a partner. Match each cubic polynomial equation with the graph of its related polynomial function. Then find *all* solutions. Make a conjecture about how you can use a graph or table of values to determine the number and types of solutions of a cubic polynomial equation.

a. $x^3 - 3x^2 + x + 5 = 0$

b. $x^3 - 2x^2 - x + 2 = 0$

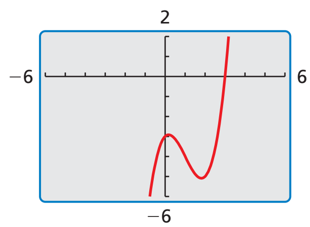
c. $x^3 - x^2 - 4x + 4 = 0$

d. $x^3 + 5x^2 + 8x + 6 = 0$

e. $x^3 - 3x^2 + x - 3 = 0$

f. $x^3 - 3x^2 + 2x = 0$

A.



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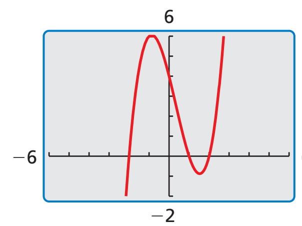

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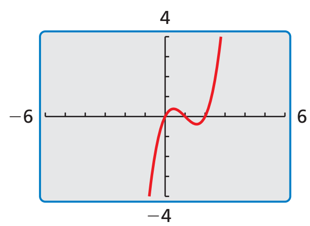

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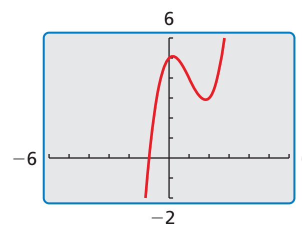

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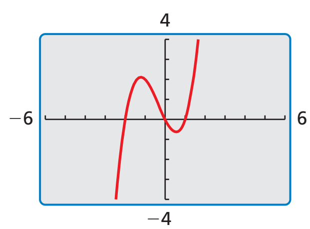

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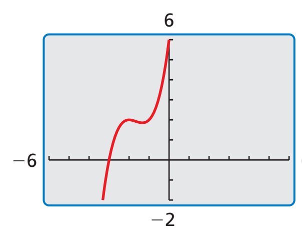


USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technology to enable you to visualize results and explore consequences.

EXPLORATION 2 Quartic Equations and Imaginary Solutions

Work with a partner. Use the graph of the related quartic function, or a table of values, to determine whether each quartic equation has imaginary solutions. Explain your reasoning. Then find *all* solutions.

a. $x^4 - 2x^3 - x^2 + 2x = 0$

b. $x^4 - 1 = 0$

c. $x^4 + x^3 - x - 1 = 0$

d. $x^4 - 3x^3 + x^2 + 3x - 2 = 0$

Communicate Your Answer

3. How can you determine whether a polynomial equation has imaginary solutions?

4. Is it possible for a cubic equation to have three imaginary solutions? Explain your reasoning.

4.6 Lesson

Core Vocabulary

complex conjugates, *p. 199*

Previous
repeated solution
degree of a polynomial
solution of an equation
zero of a function
conjugates

What You Will Learn

- Use the Fundamental Theorem of Algebra.
- Find conjugate pairs of complex zeros of polynomial functions.
- Use Descartes's Rule of Signs.

The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777–1855).

Core Concept

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

STUDY TIP

The statements "the polynomial equation $f(x) = 0$ has exactly n solutions" and "the polynomial function f has exactly n zeros" are equivalent.

The corollary to the Fundamental Theorem of Algebra also means that an nth-degree polynomial function f has exactly n zeros.

EXAMPLE 1 Finding the Number of Solutions or Zeros

a. How many solutions does the equation $x^3 + 3x^2 + 16x + 48 = 0$ have?

b. How many zeros does the function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ have?

SOLUTION

a. Because $x^3 + 3x^2 + 16x + 48 = 0$ is a polynomial equation of degree 3, it has three solutions. (The solutions are -3, $4i$, and $-4i$.)

b. Because $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ is a polynomial function of degree 4, it has four zeros. (The zeros are -2, -2, -2, and 0.)

EXAMPLE 2 Finding the Zeros of a Polynomial Function

Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$.

SOLUTION

Step 1 Find the rational zeros of f. Because f is a polynomial function of degree 5, it has five zeros. The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Using synthetic division, you can determine that -1 is a zero repeated twice and 2 is also a zero.

Step 2 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factors $x + 1$, $x + 1$, and $x - 2$ gives a quotient of $x^2 + 4$. So,

$$f(x) = (x + 1)^2(x - 2)(x^2 + 4).$$

Step 3 Find the complex zeros of f. Solving $x^2 + 4 = 0$, you get $x = \pm 2i$. This means $x^2 + 4 = (x + 2i)(x - 2i)$.

$$f(x) = (x + 1)^2(x - 2)(x + 2i)(x - 2i)$$

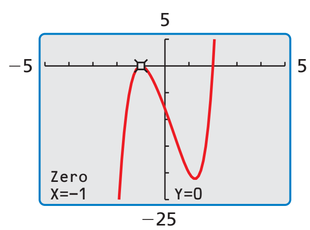
STUDY TIP

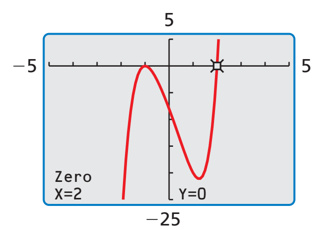
Notice that you can use imaginary numbers to write $(x^2 + 4)$ as $(x + 2i)(x - 2i)$. In general, $(a^2 + b^2) = (a + bi)(a - bi)$.

▶ From the factorization, there are five zeros. The zeros of f are

$$-1, -1, 2, -2i, \text{ and } 2i.$$

The graph of f and the real zeros are shown. Notice that only the *real* zeros appear as x-intercepts. Also, the graph of f touches the x-axis at the repeated zero $x = -1$ and crosses the x-axis at $x = 2$.





Monitoring Progress

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1. How many solutions does the equation $x^4 + 7x^2 - 144 = 0$ have?
2. How many zeros does the function $f(x) = x^3 - 5x^2 - 8x + 48$ have?

Find all zeros of the polynomial function.

3. $f(x) = x^3 + 7x^2 + 16x + 12$
4. $f(x) = x^5 - 3x^4 + 5x^3 - x^2 - 6x + 4$

Complex Conjugates

Pairs of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called **complex conjugates**. In Example 2, notice that the zeros $2i$ and $-2i$ are complex conjugates. This illustrates the next theorem.

Core Concept

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

EXAMPLE 3 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$.

SOLUTION

Because the coefficients are rational and $3 + i$ is a zero, $3 - i$ must also be a zero by the Complex Conjugates Theorem. Use the three zeros and the Factor Theorem to write $f(x)$ as a product of three factors.

$$
\begin{aligned}
f(x) &= (x - 2)[x - (3 + i)][x - (3 - i)] && \text{Write } f(x) \text{ in factored form.}\\
&= (x - 2)[(x - 3) - i][(x - 3) + i] && \text{Regroup terms.}\\
&= (x - 2)[(x - 3)^2 - i^2] && \text{Multiply.}\\
&= (x - 2)[(x^2 - 6x + 9) - (-1)] && \text{Expand binomial and use } i^2 = -1.\\
&= (x - 2)(x^2 - 6x + 10) && \text{Simplify.}\\
&= x^3 - 6x^2 + 10x - 2x^2 + 12x - 20 && \text{Multiply.}\\
&= x^3 - 8x^2 + 22x - 20 && \text{Combine like terms.}
\end{aligned}
$$

Check

You can check this result by evaluating f at each of its three zeros.

$$f(2) = (2)^3 - 8(2)^2 + 22(2) - 20 = 8 - 32 + 44 - 20 = 0 \checkmark$$

$$
\begin{aligned}
f(3 + i) &= (3 + i)^3 - 8(3 + i)^2 + 22(3 + i) - 20\\
&= 18 + 26i - 64 - 48i + 66 + 22i - 20\\
&= 0 \checkmark
\end{aligned}
$$

Because $f(3 + i) = 0$, by the Complex Conjugates Theorem $f(3 - i) = 0$. ✓

Monitoring Progress

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Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. $-1, 4i$

6. $3, 1 + i\sqrt{5}$

7. $\sqrt{2}, 1 - 3i$

8. $2, 2i, 4 - \sqrt{6}$

Descartes's Rule of Signs

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

Core Concept

Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

EXAMPLE 4 Using Descartes's Rule of Signs

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for $f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$.

SOLUTION

$$f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8.$$

The coefficients in $f(x)$ have 3 sign changes, so f has 3 or 1 positive real zero(s).

$$f(-x) = (-x)^6 - 2(-x)^5 + 3(-x)^4 - 10(-x)^3 - 6(-x)^2 - 8(-x) - 8$$
$$= x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8$$

The coefficients in $f(-x)$ have 3 sign changes, so f has 3 or 1 negative zero(s).

▶ The possible numbers of zeros for f are summarized in the table below.

Positive real zeros	Negative real zeros	Imaginary zeros	Total zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

EXAMPLE 5 Real-Life Application

A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the boat are modeled by

$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0.$$

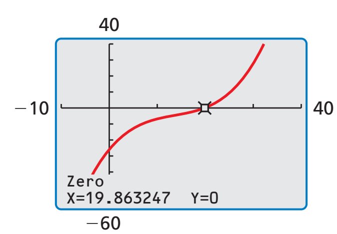
What is the tachometer reading when the boat travels 15 miles per hour?

SOLUTION

Substitute 15 for $s(x)$ in the function. You can rewrite the resulting equation as

$$0 = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0.$$

The related function to this equation is $f(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0$. By Descartes's Rule of Signs, you know f has 3 or 1 positive real zero(s). In the context of speed, negative real zeros and imaginary zeros do not make sense, so you do not need to check for them. To approximate the positive real zeros of f, use a graphing calculator. From the graph, there is 1 real zero, $x \approx 19.9$.



▶ The tachometer reading is about 1990 RPMs.

Monitoring Progress

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Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

9. $f(x) = x^3 + 9x - 25$

10. $f(x) = 3x^4 - 7x^3 + x^2 - 13x + 8$

11. **WHAT IF?** In Example 5, what is the tachometer reading when the boat travels 20 miles per hour?

4.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The expressions $5 + i$ and $5 - i$ are ____________.

2. **WRITING** How many solutions does the polynomial equation $(x + 8)^3(x - 1) = 0$ have? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, identify the number of solutions or zeros. *(See Example 1.)*

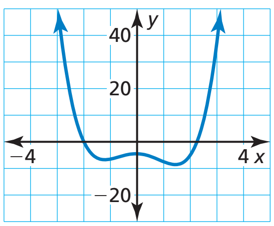
3. $x^4 + 2x^3 - 4x^2 + x = 0$
4. $5y^3 - 3y^2 + 8y = 0$
5. $9t^6 - 14t^3 + 4t - 1 = 0$
6. $f(z) = -7z^4 + z^2 - 25$
7. $g(s) = 4s^5 - s^3 + 2s^7 - 2$
8. $h(x) = 5x^4 + 7x^8 - x^{12}$

In Exercises 9–16, find all zeros of the polynomial function. *(See Example 2.)*

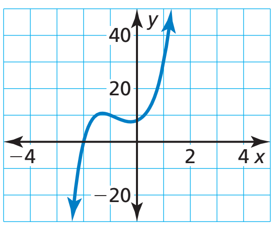
9. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$
10. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$
11. $g(x) = x^4 - 9x^2 - 4x + 12$
12. $h(x) = x^3 + 5x^2 - 4x - 20$
13. $g(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$
14. $h(x) = x^4 - x^3 + 7x^2 - 9x - 18$
15. $g(x) = x^5 + 3x^4 - 4x^3 - 2x^2 - 12x - 16$
16. $f(x) = x^5 - 20x^3 + 20x^2 - 21x + 20$

ANALYZING RELATIONSHIPS In Exercises 17–20, determine the number of imaginary zeros for the function with the given degree and graph. Explain your reasoning.

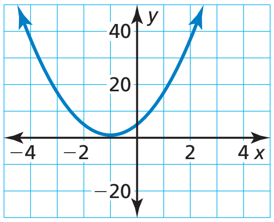
17. Degree: 4



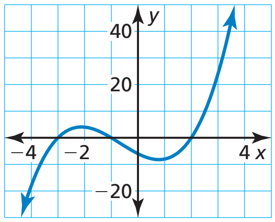
18. Degree: 5



19. Degree: 2



20. Degree: 3



In Exercises 21–28, write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. *(See Example 3.)*

21. $-5, -1, 2$
22. $-2, 1, 3$
23. $3, 4 + i$
24. $2, 5 - i$
25. $4, -\sqrt{5}$
26. $3i, 2 - i$
27. $2, 1 + i, 2 - \sqrt{3}$
28. $3, 4 + 2i, 1 + \sqrt{7}$

ERROR ANALYSIS In Exercises 29 and 30, describe and correct the error in writing a polynomial function with rational coefficients and the given zero(s).

29. Zeros: $2, 1 + i$

$$\begin{aligned} f(x) &= (x - 2)[x - (1 + i)] \\ &= x(x - 1 - i) - 2(x - 1 - i) \\ &= x^2 - x - ix - 2x + 2 + 2i \\ &= x^2 - (3 + i)x + (2 + 2i) \end{aligned}$$

30. Zero: $2 + i$

$$\begin{aligned} f(x) &= [x - (2 + i)][x + (2 + i)] \\ &= (x - 2 - i)(x + 2 + i) \\ &= x^2 + 2x + ix - 2x - 4 - 2i - ix - 2i - i^2 \\ &= x^2 - 4i - 3 \end{aligned}$$

31. **OPEN-ENDED** Write a polynomial function of degree 6 with zeros 1, 2, and $-i$. Justify your answer.

32. **REASONING** Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

In Exercises 33–40, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. *(See Example 4.)*

33. $g(x) = x^4 - x^2 - 6$

34. $g(x) = -x^3 + 5x^2 + 12$

35. $g(x) = x^3 - 4x^2 + 8x + 7$

36. $g(x) = x^5 - 2x^3 - x^2 + 6$

37. $g(x) = x^5 - 3x^3 + 8x - 10$

38. $g(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15$

39. $g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18$

40. $g(x) = x^7 + 4x^4 - 10x + 25$

41. **REASONING** Which is *not* a possible classification of zeros for $f(x) = x^5 - 4x^3 + 6x^2 + 2x - 6$? Explain.

Ⓐ three positive real zeros, two negative real zeros, and zero imaginary zeros

Ⓑ three positive real zeros, zero negative real zeros, and two imaginary zeros

Ⓒ one positive real zero, four negative real zeros, and zero imaginary zeros

Ⓓ one positive real zero, two negative real zeros, and two imaginary zeros

42. **USING STRUCTURE** Use Descartes's Rule of Signs to determine which function has at least 1 positive real zero.

Ⓐ $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

Ⓑ $f(x) = x^4 + 4x^3 + 8x^2 + 16x + 16$

Ⓒ $f(x) = -x^4 - 5x^2 - 4$

Ⓓ $f(x) = x^4 + 4x^3 + 7x^2 + 12x + 12$

43. **MODELING WITH MATHEMATICS** From 1890 to 2000, the American Indian, Eskimo, and Aleut population P (in thousands) can be modeled by the function $P = 0.004t^3 - 0.24t^2 + 4.9t + 243$, where t is the number of years since 1890. In which year did the population first reach 722,000? *(See Example 5.)*

44. **MODELING WITH MATHEMATICS** Over a period of 14 years, the number N of inland lakes infested with zebra mussels in a certain state can be modeled by

$$N = -0.0284t^4 + 0.5937t^3 - 2.464t^2 + 8.33t - 2.5$$

where t is time (in years). In which year did the number of infested inland lakes first reach 120?

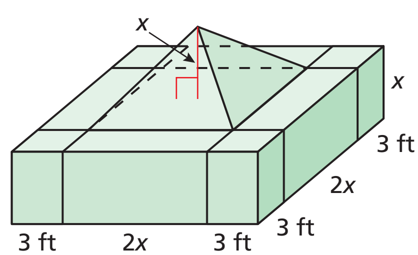
45. **MODELING WITH MATHEMATICS** For the 12 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by the function

$$R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13{,}650)$$

where t is the number of years since the store opened. In which year(s) was the revenue \$1.5 million?

46. **MAKING AN ARGUMENT** Your friend claims that $2 - i$ is a complex zero of the polynomial function $f(x) = x^3 - 2x^2 + 2x + 5i$, but that its conjugate is *not* a zero. You claim that both $2 - i$ and its conjugate *must* be zeros by the Complex Conjugates Theorem. Who is correct? Justify your answer.

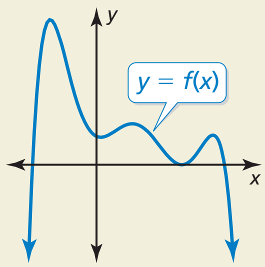
47. **MATHEMATICAL CONNECTIONS** A solid monument with the dimensions shown is to be built using 1000 cubic feet of marble. What is the value of x?



48. **THOUGHT PROVOKING** Write and graph a polynomial function of degree 5 that has all positive or negative real zeros. Label each x-intercept. Then write the function in standard form.

49. **WRITING** The graph of the constant polynomial function $f(x) = 2$ is a line that does not have any x-intercepts. Does the function contradict the Fundamental Theorem of Algebra? Explain.

50. **HOW DO YOU SEE IT?** The graph represents a polynomial function of degree 6.



a. How many positive real zeros does the function have? negative real zeros? imaginary zeros?

b. Use Descartes's Rule of Signs and your answers in part (a) to describe the possible sign changes in the coefficients of $f(x)$.

51. **FINDING A PATTERN** Use a graphing calculator to graph the function $f(x) = (x + 3)^n$ for $n = 2, 3, 4, 5, 6$, and 7.

a. Compare the graphs when n is even and n is odd.

b. Describe the behavior of the graph near the zero $x = -3$ as n increases.

c. Use your results from parts (a) and (b) to describe the behavior of the graph of $g(x) = (x - 4)^{20}$ near $x = 4$.

52. **DRAWING CONCLUSIONS** Find the zeros of each function.

$$f(x) = x^2 - 5x + 6$$
$$g(x) = x^3 - 7x + 6$$
$$h(x) = x^4 + 2x^3 + x^2 + 8x - 12$$
$$k(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$$

a. Describe the relationship between the sum of the zeros of a polynomial function and the coefficients of the polynomial function.

b. Describe the relationship between the product of the zeros of a polynomial function and the coefficients of the polynomial function.

53. **PROBLEM SOLVING** You want to save money so you can buy a used car in four years. At the end of each summer, you deposit \$1000 earned from summer jobs into your bank account. The table shows the value of your deposits over the four-year period. In the table, g is the growth factor $1 + r$, where r is the annual interest rate expressed as a decimal.

Deposit	Year 1	Year 2	Year 3	Year 4
1st Deposit	1000	$1000g$	$1000g^2$	$1000g^3$
2nd Deposit	–	1000		
3rd Deposit	–	–	1000	
4th Deposit	–	–	–	1000

a. Copy and complete the table.

b. Write a polynomial function that gives the value v of your account at the end of the fourth summer in terms of g.

c. You want to buy a car that costs about \$4300. What growth factor do you need to obtain this amount? What annual interest rate do you need?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Describe the transformation of $f(x) = x^2$ represented by g. Then graph each function. *(Section 2.1)*

54. $g(x) = -3x^2$

55. $g(x) = (x - 4)^2 + 6$

56. $g(x) = -(x - 1)^2$

57. $g(x) = 5(x + 4)^2$

Write a function g whose graph represents the indicated transformation of the graph of f. *(Sections 1.2 and 2.1)*

58. $f(x) = x$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y-axis

59. $f(x) = |x + 1| - 3$; horizontal stretch by a factor of 9

60. $f(x) = x^2$; reflection in the x-axis, followed by a translation 2 units right and 7 units up