

## 2.3 Focus of a Parabola



Learning Standards  
HSF-IF.B.4  
HSF-IF.C.7c  
HSG-GPE.A.2

### CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build logical progressions of statements to explore the truth of your conjectures.

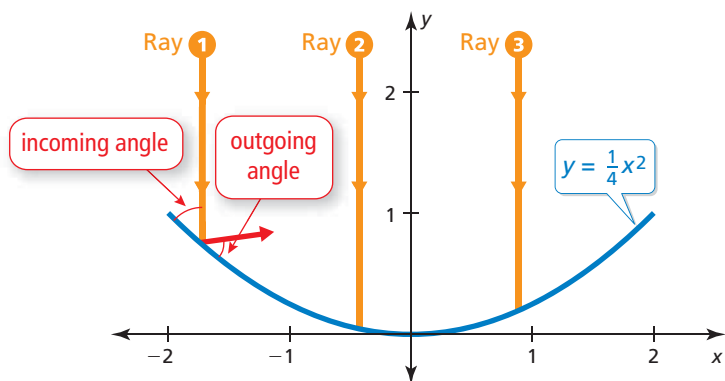
### Essential Question

What is the focus of a parabola?

#### EXPLORATION 1 Analyzing Satellite Dishes

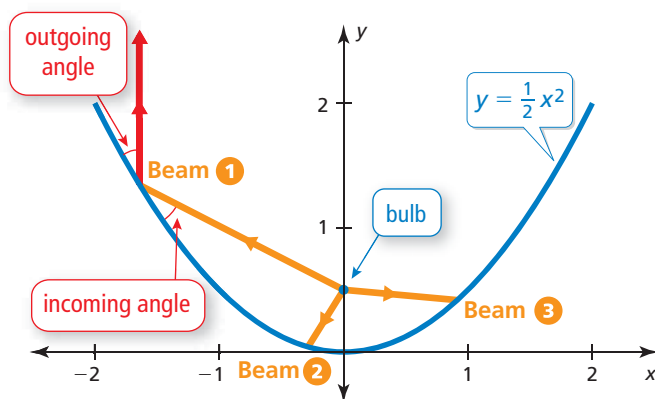
**Work with a partner.** Vertical rays enter a satellite dish whose cross section is a parabola. When the rays hit the parabola, they reflect at the same angle at which they entered. (See Ray 1 in the figure.)

- Draw the reflected rays so that they intersect the  $y$ -axis.
- What do the reflected rays have in common?
- The optimal location for the receiver of the satellite dish is at a point called the *focus* of the parabola. Determine the location of the focus. Explain why this makes sense in this situation.



#### EXPLORATION 2 Analyzing Spotlights

**Work with a partner.** Beams of light are coming from the bulb in a spotlight, located at the focus of the parabola. When the beams hit the parabola, they reflect at the same angle at which they hit. (See Beam 1 in the figure.) Draw the reflected beams. What do they have in common? Would you consider this to be the optimal result? Explain.



### Communicate Your Answer

- What is the focus of a parabola?
- Describe some of the properties of the focus of a parabola.

## 2.3 Lesson

### Core Vocabulary

focus, p. 68  
directrix, p. 68

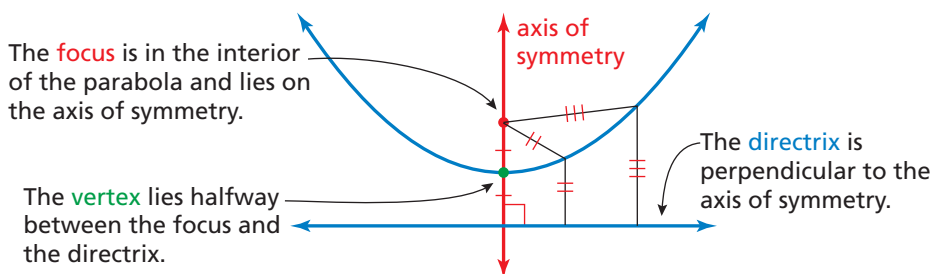
**Previous**  
perpendicular  
Distance Formula  
congruent

## What You Will Learn

- ▶ Explore the focus and the directrix of a parabola.
- ▶ Write equations of parabolas.
- ▶ Solve real-life problems.

### Exploring the Focus and Directrix

Previously, you learned that the graph of a quadratic function is a parabola that opens up or down. A parabola can also be defined as the set of all points  $(x, y)$  in a plane that are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**.



### STUDY TIP

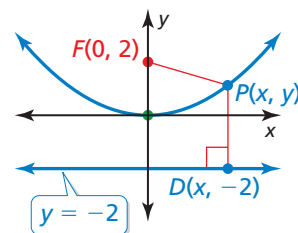
The distance from a point to a line is defined as the length of the perpendicular segment from the point to the line.

### EXAMPLE 1 Using the Distance Formula to Write an Equation

Use the Distance Formula to write an equation of the parabola with focus  $F(0, 2)$  and directrix  $y = -2$ .

#### SOLUTION

Notice the line segments drawn from point  $F$  to point  $P$  and from point  $P$  to point  $D$ . By the definition of a parabola, these line segments must be congruent.



Definition of a parabola

Distance Formula

Substitute for  $x_1, y_1, x_2,$  and  $y_2$ .

Simplify.

Square each side.

Expand.

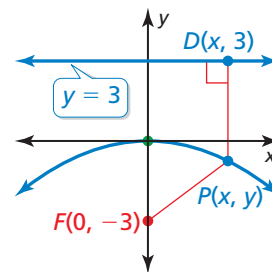
Combine like terms.

Divide each side by 8.

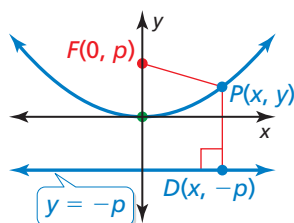
$$\begin{aligned}
 PD &= PF \\
 \sqrt{(x - x_1)^2 + (y - y_1)^2} &= \sqrt{(x - x_2)^2 + (y - y_2)^2} \\
 \sqrt{(x - 0)^2 + (y - (-2))^2} &= \sqrt{(x - 0)^2 + (y - 2)^2} \\
 \sqrt{(y + 2)^2} &= \sqrt{x^2 + (y - 2)^2} \\
 (y + 2)^2 &= x^2 + (y - 2)^2 \\
 y^2 + 4y + 4 &= x^2 + y^2 - 4y + 4 \\
 8y &= x^2 \\
 y &= \frac{1}{8}x^2
 \end{aligned}$$

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1. Use the Distance Formula to write an equation of the parabola with focus  $F(0, -3)$  and directrix  $y = 3$ .



You can derive the equation of a parabola that opens up or down with vertex  $(0, 0)$ , focus  $(0, p)$ , and directrix  $y = -p$  using the procedure in Example 1.



$$\begin{aligned} \sqrt{(x-x)^2 + (y-(-p))^2} &= \sqrt{(x-0)^2 + (y-p)^2} \\ (y+p)^2 &= x^2 + (y-p)^2 \\ y^2 + 2py + p^2 &= x^2 + y^2 - 2py + p^2 \\ 4py &= x^2 \\ y &= \frac{1}{4p}x^2 \end{aligned}$$

## LOOKING FOR STRUCTURE

Notice that  $y = \frac{1}{4p}x^2$  is of the form  $y = ax^2$ . So, changing the value of  $p$  vertically stretches or shrinks the parabola.

The focus and directrix each lie  $|p|$  units from the vertex. Parabolas can also open left or right, in which case the equation has the form  $x = \frac{1}{4p}y^2$  when the vertex is  $(0, 0)$ .

## Core Concept

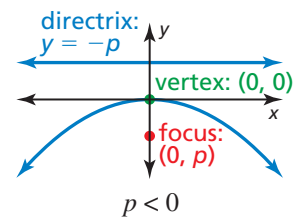
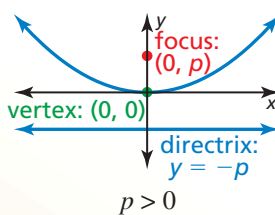
### Standard Equations of a Parabola with Vertex at the Origin

#### Vertical axis of symmetry ( $x = 0$ )

Equation:  $y = \frac{1}{4p}x^2$

Focus:  $(0, p)$

Directrix:  $y = -p$

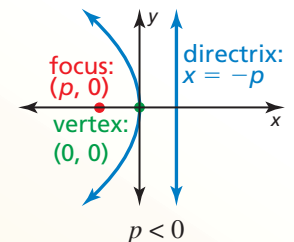
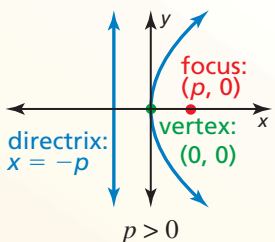


#### Horizontal axis of symmetry ( $y = 0$ )

Equation:  $x = \frac{1}{4p}y^2$

Focus:  $(p, 0)$

Directrix:  $x = -p$



## STUDY TIP

Notice that parabolas opening left or right do *not* represent functions.

## EXAMPLE 2 Graphing an Equation of a Parabola

Identify the focus, directrix, and axis of symmetry of  $-4x = y^2$ . Graph the equation.

### SOLUTION

**Step 1** Rewrite the equation in standard form.

$$-4x = y^2 \quad \text{Write the original equation.}$$

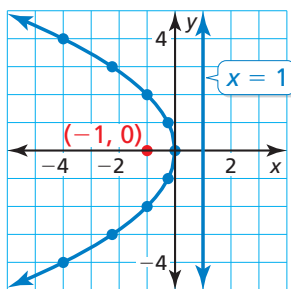
$$x = -\frac{1}{4}y^2 \quad \text{Divide each side by } -4.$$

**Step 2** Identify the focus, directrix, and axis of symmetry. The equation has the form

$x = \frac{1}{4p}y^2$ , where  $p = -1$ . The focus is  $(p, 0)$ , or  $(-1, 0)$ . The directrix is  $x = -p$ , or  $x = 1$ . Because  $y$  is squared, the axis of symmetry is the  $x$ -axis.

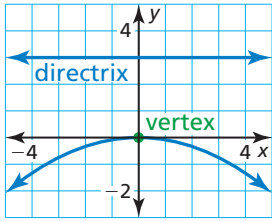
**Step 3** Use a table of values to graph the equation. Notice that it is easier to substitute  $y$ -values and solve for  $x$ . Opposite  $y$ -values result in the same  $x$ -value.

$y$	0	$\pm 1$	$\pm 2$	$\pm 3$	$\pm 4$
$x$	0	-0.25	-1	-2.25	-4



## Writing Equations of Parabolas

### EXAMPLE 3 Writing an Equation of a Parabola



Write an equation of the parabola shown.

#### SOLUTION

Because the vertex is at the origin and the axis of symmetry is vertical, the equation has the form  $y = \frac{1}{4p}x^2$ . The directrix is  $y = -p = 3$ , so  $p = -3$ . Substitute  $-3$  for  $p$  to write an equation of the parabola.

$$y = \frac{1}{4(-3)}x^2 = -\frac{1}{12}x^2$$

► So, an equation of the parabola is  $y = -\frac{1}{12}x^2$ .

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Identify the focus, directrix, and axis of symmetry of the parabola. Then graph the equation.

2.  $y = 0.5x^2$

3.  $-y = x^2$

4.  $y^2 = 6x$

Write an equation of the parabola with vertex at  $(0, 0)$  and the given directrix or focus.

5. directrix:  $x = -3$

6. focus:  $(-2, 0)$

7. focus:  $(0, \frac{3}{2})$

The vertex of a parabola is not always at the origin. As in previous transformations, adding a value to the input or output of a function translates its graph.

## Core Concept

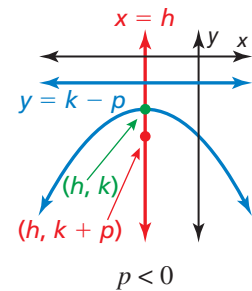
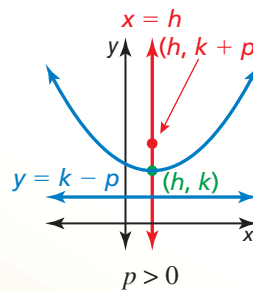
### Standard Equations of a Parabola with Vertex at $(h, k)$

#### Vertical axis of symmetry ( $x = h$ )

Equation:  $y = \frac{1}{4p}(x - h)^2 + k$

Focus:  $(h, k + p)$

Directrix:  $y = k - p$

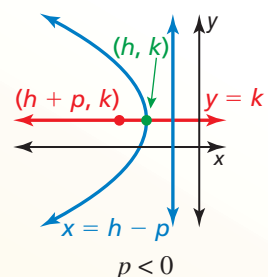
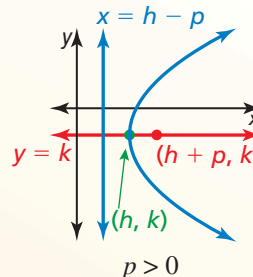


#### Horizontal axis of symmetry ( $y = k$ )

Equation:  $x = \frac{1}{4p}(y - k)^2 + h$

Focus:  $(h + p, k)$

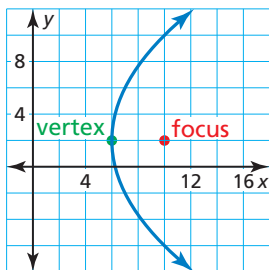
Directrix:  $x = h - p$



### STUDY TIP

The standard form for a vertical axis of symmetry looks like vertex form. To remember the standard form for a horizontal axis of symmetry, switch  $x$  and  $y$ , and  $h$  and  $k$ .

### EXAMPLE 4 Writing an Equation of a Translated Parabola



Write an equation of the parabola shown.

#### SOLUTION

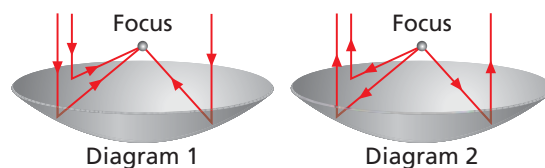
Because the vertex is not at the origin and the axis of symmetry is horizontal, the equation has the form  $x = \frac{1}{4p}(y - k)^2 + h$ . The vertex  $(h, k)$  is  $(6, 2)$  and the focus  $(h + p, k)$  is  $(10, 2)$ , so  $h = 6$ ,  $k = 2$ , and  $p = 4$ . Substitute these values to write an equation of the parabola.

$$x = \frac{1}{4(4)}(y - 2)^2 + 6 = \frac{1}{16}(y - 2)^2 + 6$$

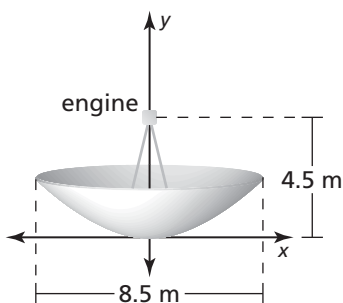
► So, an equation of the parabola is  $x = \frac{1}{16}(y - 2)^2 + 6$ .

### Solving Real-Life Problems

Parabolic reflectors have cross sections that are parabolas. Incoming sound, light, or other energy that arrives at a parabolic reflector parallel to the axis of symmetry is directed to the focus (Diagram 1). Similarly, energy that is emitted from the focus of a parabolic reflector and then strikes the reflector is directed parallel to the axis of symmetry (Diagram 2).



### EXAMPLE 5 Solving a Real-Life Problem



An electricity-generating dish uses a parabolic reflector to concentrate sunlight onto a high-frequency engine located at the focus of the reflector. The sunlight heats helium to  $650^\circ\text{C}$  to power the engine. Write an equation that represents the cross section of the dish shown with its vertex at  $(0, 0)$ . What is the depth of the dish?

#### SOLUTION

Because the vertex is at the origin, and the axis of symmetry is vertical, the equation has the form  $y = \frac{1}{4p}x^2$ . The engine is at the focus, which is 4.5 meters above the vertex. So,  $p = 4.5$ . Substitute 4.5 for  $p$  to write the equation.

$$y = \frac{1}{4(4.5)}x^2 = \frac{1}{18}x^2$$

The depth of the dish is the  $y$ -value at the dish's outside edge. The dish extends  $\frac{8.5}{2} = 4.25$  meters to either side of the vertex  $(0, 0)$ , so find  $y$  when  $x = 4.25$ .

$$y = \frac{1}{18}(4.25)^2 \approx 1$$

► The depth of the dish is about 1 meter.

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- Write an equation of a parabola with vertex  $(-1, 4)$  and focus  $(-1, 2)$ .
- A parabolic microwave antenna is 16 feet in diameter. Write an equation that represents the cross section of the antenna with its vertex at  $(0, 0)$  and its focus 10 feet to the right of the vertex. What is the depth of the antenna?

## 2.3 Exercises

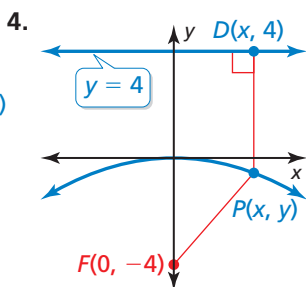
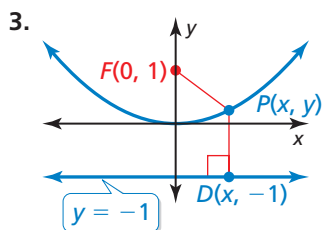
Dynamic Solutions available at [BigIdeasMath.com](http://BigIdeasMath.com)

### Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** A parabola is the set of all points in a plane equidistant from a fixed point called the \_\_\_\_\_ and a fixed line called the \_\_\_\_\_.
- WRITING** Explain how to find the coordinates of the focus of a parabola with vertex  $(0, 0)$  and directrix  $y = 5$ .

### Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, use the Distance Formula to write an equation of the parabola. (See Example 1.)

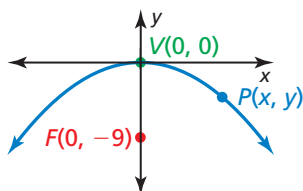


- focus:  $(0, -2)$   
directrix:  $y = 2$
- vertex:  $(0, 0)$   
directrix:  $y = -6$
- vertex:  $(0, 0)$   
focus:  $(0, 5)$
- vertex:  $(0, 0)$   
focus:  $(0, -10)$
- directrix:  $y = 7$   
focus:  $(0, -7)$
- vertex:  $(0, 0)$   
focus:  $(0, 5)$
- vertex:  $(0, 0)$   
directrix:  $y = -9$

11. **ANALYZING RELATIONSHIPS** Which of the given characteristics describe parabolas that open down? Explain your reasoning.

- |  |  |
|--|--|
| (A) focus: $(0, -6)$<br>directrix: $y = 6$ | (B) focus: $(0, -2)$<br>directrix: $y = 2$ |
| (C) focus: $(0, 6)$<br>directrix: $y = -6$ | (D) focus: $(0, -1)$<br>directrix: $y = 1$ |

12. **REASONING** Which of the following are possible coordinates of the point  $P$  in the graph shown? Explain.

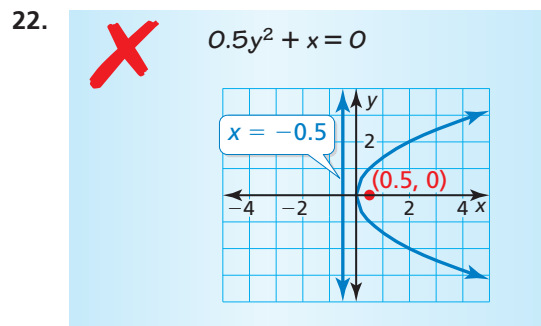
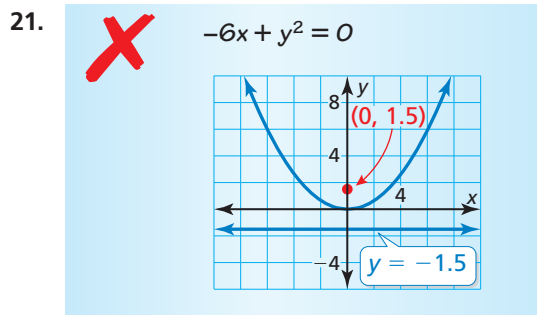


- |                         |                         |                          |
|-------------------------|-------------------------|--------------------------|
| (A) $(-6, -1)$          | (B) $(3, -\frac{1}{4})$ | (C) $(4, -\frac{4}{9})$  |
| (D) $(1, \frac{1}{36})$ | (E) $(6, -1)$           | (F) $(2, -\frac{1}{18})$ |

In Exercises 13–20, identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation. (See Example 2.)

- |                            |                            |
|----------------------------|----------------------------|
| 13. $y = \frac{1}{8}x^2$   | 14. $y = -\frac{1}{12}x^2$ |
| 15. $x = -\frac{1}{20}y^2$ | 16. $x = \frac{1}{24}y^2$  |
| 17. $y^2 = 16x$            | 18. $-x^2 = 48y$           |
| 19. $6x^2 + 3y = 0$        | 20. $8x^2 - y = 0$         |

**ERROR ANALYSIS** In Exercises 21 and 22, describe and correct the error in graphing the parabola.

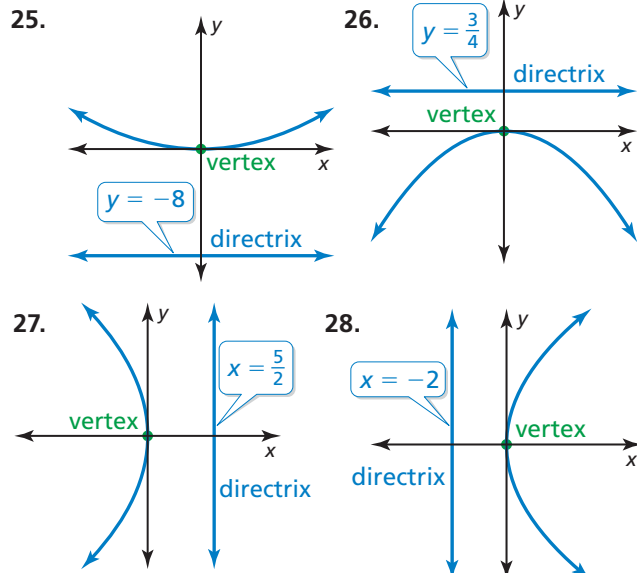


23. **ANALYZING EQUATIONS** The cross section (with units in inches) of a parabolic satellite dish can be modeled by the equation  $y = \frac{1}{38}x^2$ . How far is the receiver from the vertex of the cross section? Explain.

- 24. ANALYZING EQUATIONS** The cross section (with units in inches) of a parabolic spotlight can be modeled by the equation  $x = \frac{1}{20}y^2$ . How far is the bulb from the vertex of the cross section? Explain.



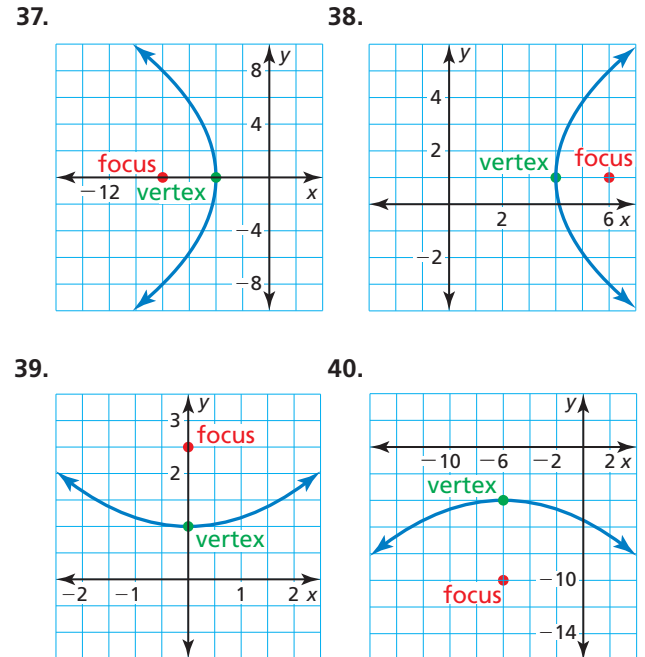
In Exercises 25–28, write an equation of the parabola shown. (See Example 3.)



In Exercises 29–36, write an equation of the parabola with the given characteristics.

- |   |   |
|---|---|
| <b>29.</b> focus: $(3, 0)$<br>directrix: $x = -3$                     | <b>30.</b> focus: $(\frac{2}{3}, 0)$<br>directrix: $x = -\frac{2}{3}$ |
| <b>31.</b> directrix: $x = -10$<br>vertex: $(0, 0)$                   | <b>32.</b> directrix: $y = \frac{8}{3}$<br>vertex: $(0, 0)$           |
| <b>33.</b> focus: $(0, -\frac{5}{3})$<br>directrix: $y = \frac{5}{3}$ | <b>34.</b> focus: $(0, \frac{5}{4})$<br>directrix: $y = -\frac{5}{4}$ |
| <b>35.</b> focus: $(0, \frac{6}{7})$<br>vertex: $(0, 0)$              | <b>36.</b> focus: $(-\frac{4}{5}, 0)$<br>vertex: $(0, 0)$             |

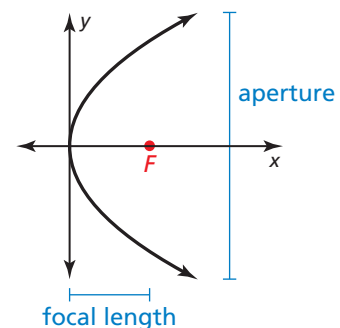
In Exercises 37–40, write an equation of the parabola shown. (See Example 4.)



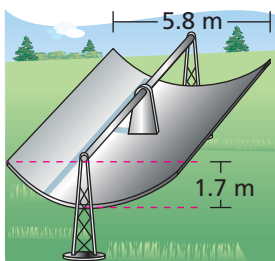
In Exercises 41–46, identify the vertex, focus, directrix, and axis of symmetry of the parabola. Describe the transformations of the graph of the standard equation with vertex  $(0, 0)$ .

- 41.**  $y = \frac{1}{8}(x - 3)^2 + 2$       **42.**  $y = -\frac{1}{4}(x + 2)^2 + 1$   
**43.**  $x = \frac{1}{16}(y - 3)^2 + 1$       **44.**  $y = (x + 3)^2 - 5$   
**45.**  $x = -3(y + 4)^2 + 2$       **46.**  $x = 4(y + 5)^2 - 1$

- 47. MODELING WITH MATHEMATICS** Scientists studying dolphin echolocation simulate the projection of a bottlenose dolphin's clicking sounds using computer models. The models originate the sounds at the focus of a parabolic reflector. The parabola in the graph shows the cross section of the reflector with focal length of 1.3 inches and aperture width of 8 inches. Write an equation to represent the cross section of the reflector. What is the depth of the reflector? (See Example 5.)

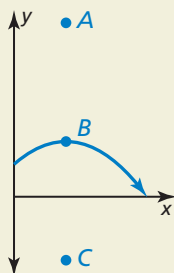


48. **MODELING WITH MATHEMATICS** Solar energy can be concentrated using long troughs that have a parabolic cross section as shown in the figure. Write an equation to represent the cross section of the trough. What are the domain and range in this situation? What do they represent?



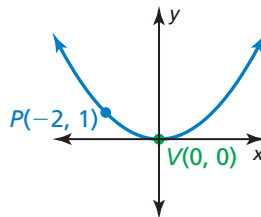
49. **ABSTRACT REASONING** As  $|p|$  increases, how does the width of the graph of the equation  $y = \frac{1}{4p}x^2$  change? Explain your reasoning.

50. **HOW DO YOU SEE IT?** The graph shows the path of a volleyball served from an initial height of 6 feet as it travels over a net.



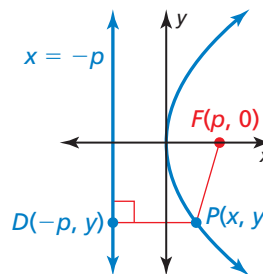
- Label the vertex, focus, and a point on the directrix.
- An underhand serve follows the same parabolic path but is hit from a height of 3 feet. How does this affect the focus? the directrix?

51. **CRITICAL THINKING** The distance from point  $P$  to the directrix is 2 units. Write an equation of the parabola.

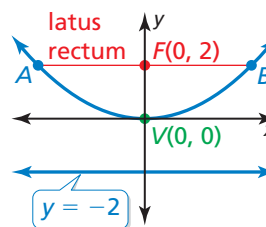


52. **THOUGHT PROVOKING** Two parabolas have the same focus  $(a, b)$  and focal length of 2 units. Write an equation of each parabola. Identify the directrix of each parabola.

53. **REPEATED REASONING** Use the Distance Formula to derive the equation of a parabola that opens to the right with vertex  $(0, 0)$ , focus  $(p, 0)$ , and directrix  $x = -p$ .



54. **PROBLEM SOLVING** The *latus rectum* of a parabola is the line segment that is parallel to the directrix, passes through the focus, and has endpoints that lie on the parabola. Find the length of the latus rectum of the parabola shown.



## Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Write an equation of the line that passes through the points. (Section 1.3)

55.  $(1, -4), (2, -1)$     56.  $(-3, 12), (0, 6)$     57.  $(3, 1), (-5, 5)$     58.  $(2, -1), (0, 1)$

Use a graphing calculator to find an equation for the line of best fit. (Section 1.3)

59.

$x$	0	3	6	7	11
$y$	4	9	24	29	46

60.

$x$	0	5	10	12	16
$y$	18	15	9	7	2