3.3 Completing the Square

Essential Question

How can you complete the square for a quadratic expression?

COMMON CORE

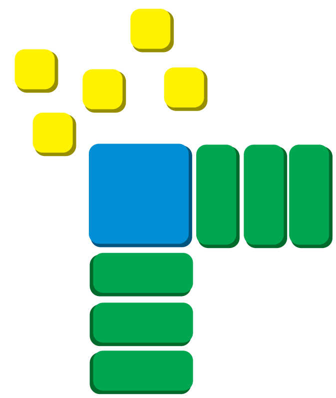
Learning Standards
HSN-CN.C.7
HSA-REI.B.4b
HSF-IF.C.8a

EXPLORATION 1 Using Algebra Tiles to Complete the Square

Work with a partner. Use algebra tiles to complete the square for the expression

$x^2 + 6x.$

a. You can model $x^2 + 6x$ using one x^2-tile and six x-tiles. Arrange the tiles in a square. Your arrangement will be incomplete in one of the corners.



b. How many 1-tiles do you need to complete the square?

c. Find the value of c so that the expression

$x^2 + 6x + c$

is a perfect square trinomial.

d. Write the expression in part (c) as the square of a binomial.

EXPLORATION 2 Drawing Conclusions

Work with a partner.

a. Use the method outlined in Exploration 1 to complete the table.

Expression	Value of c needed to complete the square	Expression written as a binomial squared
$x^2 + 2x + c$		
$x^2 + 4x + c$		
$x^2 + 8x + c$		
$x^2 + 10x + c$		

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

b. Look for patterns in the last column of the table. Consider the general statement $x^2 + bx + c = (x + d)^2$. How are d and b related in each case? How are c and d related in each case?

c. How can you obtain the values in the second column directly from the coefficients of x in the first column?

Communicate Your Answer

3. How can you complete the square for a quadratic expression?

4. Describe how you can solve the quadratic equation $x^2 + 6x = 1$ by completing the square.

3.3 Lesson

Core Vocabulary

completing the square, *p. 112*

Previous
perfect square trinomial
vertex form

What You Will Learn

- Solve quadratic equations using square roots.
- Solve quadratic equations by completing the square.
- Write quadratic functions in vertex form.

Solving Quadratic Equations Using Square Roots

Previously, you have solved equations of the form $u^2 = d$ by taking the square root of each side. This method also works when one side of an equation is a perfect square trinomial and the other side is a constant.

EXAMPLE 1 Solving a Quadratic Equation Using Square Roots

Solve $x^2 - 16x + 64 = 100$ using square roots.

SOLUTION

$x^2 - 16x + 64 = 100$ Write the equation.

$(x - 8)^2 = 100$ Write the left side as a binomial squared.

$x - 8 = \pm 10$ Take square root of each side.

$x = 8 \pm 10$ Add 8 to each side.

So, the solutions are $x = 8 + 10 = 18$ and $x = 8 - 10 = -2$.

ANOTHER WAY

You can also solve the equation by writing it in standard form as $x^2 - 16x - 36 = 0$ and factoring.

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Solve the equation using square roots. Check your solution(s).

1. $x^2 + 4x + 4 = 36$ **2.** $x^2 - 6x + 9 = 1$ **3.** $x^2 - 22x + 121 = 81$

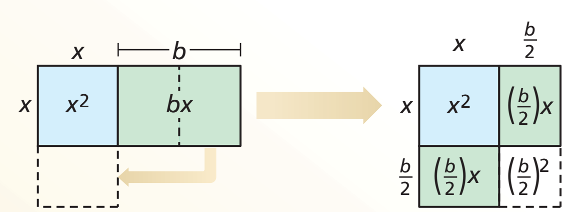
In Example 1, the expression $x^2 - 16x + 64$ is a perfect square trinomial because it equals $(x - 8)^2$. Sometimes you need to add a term to an expression $x^2 + bx$ to make it a perfect square trinomial. This process is called **completing the square**.

Core Concept

Completing the Square

Words To complete the square for the expression $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.

Diagrams In each diagram, the combined area of the shaded regions is $x^2 + bx$. Adding $\left(\frac{b}{2}\right)^2$ completes the square in the second diagram.



Algebra $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = \left(x + \frac{b}{2}\right)^2$

Solving Quadratic Equations by Completing the Square

EXAMPLE 2 Making a Perfect Square Trinomial

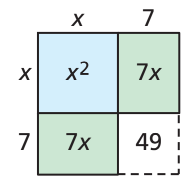
Find the value of c that makes $x^2 + 14x + c$ a perfect square trinomial. Then write the expression as the square of a binomial.

SOLUTION

Step 1 Find half the coefficient of x. $\quad \frac{14}{2} = 7$

Step 2 Square the result of Step 1. $\quad 7^2 = 49$

Step 3 Replace c with the result of Step 2. $\quad x^2 + 14x + 49$



▶ The expression $x^2 + 14x + c$ is a perfect square trinomial when $c = 49$. Then $x^2 + 14x + 49 = (x + 7)(x + 7) = (x + 7)^2$.

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Find the value of c that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial.

4. $x^2 + 8x + c$ **5.** $x^2 - 2x + c$ **6.** $x^2 - 9x + c$

The method of completing the square can be used to solve *any* quadratic equation. When you complete the square as part of solving an equation, you must add the same number to *both* sides of the equation.

LOOKING FOR STRUCTURE

Notice you cannot solve the equation by factoring because $x^2 - 10x + 7$ is not factorable as a product of binomials.

EXAMPLE 3 Solving $ax^2 + bx + c = 0$ when $a = 1$

Solve $x^2 - 10x + 7 = 0$ by completing the square.

SOLUTION

$x^2 - 10x + 7 = 0$ Write the equation.

$x^2 - 10x = -7$ Write left side in the form $x^2 + bx$.

$x^2 - 10x + 25 = -7 + 25$ Add $\left(\frac{b}{2}\right)^2 = \left(\frac{-10}{2}\right)^2 = 25$ to each side.

$(x - 5)^2 = 18$ Write left side as a binomial squared.

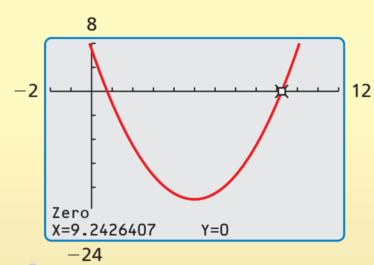
$x - 5 = \pm\sqrt{18}$ Take square root of each side.

$x = 5 \pm \sqrt{18}$ Add 5 to each side.

$x = 5 \pm 3\sqrt{2}$ Simplify radical.

▶ The solutions are $x = 5 + 3\sqrt{2}$ and $x = 5 - 3\sqrt{2}$. You can check this by graphing $y = x^2 - 10x + 7$. The x-intercepts are about $9.24 \approx 5 + 3\sqrt{2}$ and $0.76 \approx 5 - 3\sqrt{2}$.

Check



EXAMPLE 4 Solving $ax^2 + bx + c = 0$ when $a \neq 1$

Solve $3x^2 + 12x + 15 = 0$ by completing the square.

SOLUTION

The coefficient a is not 1, so you must first divide each side of the equation by a.

$3x^2 + 12x + 15 = 0$ — Write the equation.

$x^2 + 4x + 5 = 0$ — Divide each side by 3.

$x^2 + 4x = -5$ — Write left side in the form $x^2 + bx$.

$x^2 + 4x + 4 = -5 + 4$ — Add $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$ to each side.

$(x + 2)^2 = -1$ — Write left side as a binomial squared.

$x + 2 = \pm\sqrt{-1}$ — Take square root of each side.

$x = -2 \pm \sqrt{-1}$ — Subtract 2 from each side.

$x = -2 \pm i$ — Write in terms of i.

▶ The solutions are $x = -2 + i$ and $x = -2 - i$.

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Solve the equation by completing the square.

7. $x^2 - 4x + 8 = 0$

8. $x^2 + 8x - 5 = 0$

9. $-3x^2 - 18x - 6 = 0$

10. $4x^2 + 32x = -68$

11. $6x(x + 2) = -42$

12. $2x(x - 2) = 200$

Writing Quadratic Functions in Vertex Form

Recall that the vertex form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the vertex of the graph of the function. You can write a quadratic function in vertex form by completing the square.

EXAMPLE 5 Writing a Quadratic Function in Vertex Form

Write $y = x^2 - 12x + 18$ in vertex form. Then identify the vertex.

SOLUTION

$y = x^2 - 12x + 18$ — Write the function.

$y + ? = (x^2 - 12x + ?) + 18$ — Prepare to complete the square.

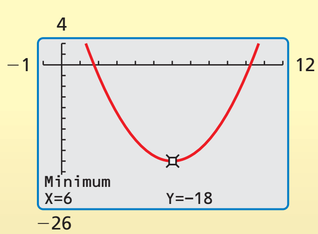
$y + 36 = (x^2 - 12x + 36) + 18$ — Add $\left(\frac{b}{2}\right)^2 = \left(\frac{-12}{2}\right)^2 = 36$ to each side.

$y + 36 = (x - 6)^2 + 18$ — Write $x^2 - 12x + 36$ as a binomial squared.

$y = (x - 6)^2 - 18$ — Solve for y.

▶ The vertex form of the function is $y = (x - 6)^2 - 18$. The vertex is $(6, -18)$.

Check



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Write the quadratic function in vertex form. Then identify the vertex.

13. $y = x^2 - 8x + 18$

14. $y = x^2 + 6x + 4$

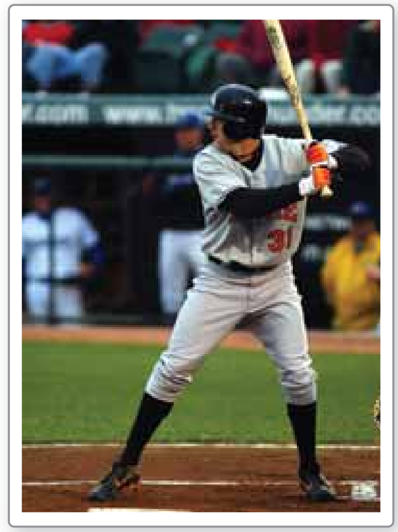
15. $y = x^2 - 2x - 6$

EXAMPLE 6 Modeling with Mathematics

The height y (in feet) of a baseball t seconds after it is hit can be modeled by the function

$$y = -16t^2 + 96t + 3.$$

Find the maximum height of the baseball. How long does the ball take to hit the ground?



SOLUTION

1. **Understand the Problem** You are given a quadratic function that represents the height of a ball. You are asked to determine the maximum height of the ball and how long it is in the air.

2. **Make a Plan** Write the function in vertex form to identify the maximum height. Then find and interpret the zeros to determine how long the ball takes to hit the ground.

3. **Solve the Problem** Write the function in vertex form by completing the square.

$y = -16t^2 + 96t + 3$	Write the function.
$y = -16(t^2 - 6t) + 3$	Factor -16 from first two terms.
$y + ? = -16(t^2 - 6t + ?) + 3$	Prepare to complete the square.
$y + (-16)(9) = -16(t^2 - 6t + 9) + 3$	Add $(-16)(9)$ to each side.
$y - 144 = -16(t - 3)^2 + 3$	Write $t^2 - 6t + 9$ as a binomial squared.
$y = -16(t - 3)^2 + 147$	Solve for y.

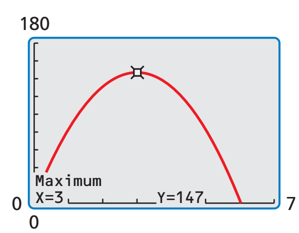
The vertex is (3, 147). Find the zeros of the function.

$0 = -16(t - 3)^2 + 147$	Substitute 0 for y.
$-147 = -16(t - 3)^2$	Subtract 147 from each side.
$9.1875 = (t - 3)^2$	Divide each side by -16.
$\pm\sqrt{9.1875} = t - 3$	Take square root of each side.
$3 \pm \sqrt{9.1875} = t$	Add 3 to each side.

Reject the negative solution, $3 - \sqrt{9.1875} \approx -0.03$, because time must be positive.

▶ So, the maximum height of the ball is 147 feet, and it takes $3 + \sqrt{9.1875} \approx 6$ seconds for the ball to hit the ground.

4. **Look Back** The vertex indicates that the maximum height of 147 feet occurs when $t = 3$. This makes sense because the graph of the function is parabolic with zeros near $t = 0$ and $t = 6$. You can use a graph to check the maximum height.



ANOTHER WAY

You can use the coefficients of the original function $y = f(x)$ to find the maximum height.

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{96}{2(-16)}\right)$$
$$= f(3)$$
$$= 147$$

LOOKING FOR STRUCTURE

You could write the zeros as $3 \pm \frac{7\sqrt{3}}{4}$, but it is easier to recognize that $3 - \sqrt{9.1875}$ is negative because $\sqrt{9.1875}$ is greater than 3.

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16. **WHAT IF?** The height of the baseball can be modeled by $y = -16t^2 + 80t + 2$. Find the maximum height of the baseball. How long does the ball take to hit the ground?

3.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What must you add to the expression $x^2 + bx$ to complete the square?

2. **COMPLETE THE SENTENCE** The trinomial $x^2 - 6x + 9$ is a ____ because it equals ____.

Monitoring Progress and Modeling with Mathematics

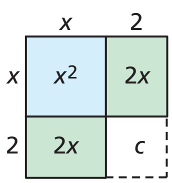
In Exercises 3–10, solve the equation using square roots. Check your solution(s). *(See Example 1.)*

3. $x^2 - 8x + 16 = 25$
4. $r^2 - 10r + 25 = 1$
5. $x^2 - 18x + 81 = 5$
6. $m^2 + 8m + 16 = 45$
7. $y^2 - 24y + 144 = -100$
8. $x^2 - 26x + 169 = -13$
9. $4w^2 + 4w + 1 = 75$
10. $4x^2 - 8x + 4 = 1$

In Exercises 11–20, find the value of c that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial. *(See Example 2.)*

11. $x^2 + 10x + c$
12. $x^2 + 20x + c$
13. $y^2 - 12y + c$
14. $t^2 - 22t + c$
15. $x^2 - 6x + c$
16. $x^2 + 24x + c$
17. $z^2 - 5z + c$
18. $x^2 + 9x + c$
19. $w^2 + 13w + c$
20. $s^2 - 26s + c$

In Exercises 21–24, find the value of c. Then write an expression represented by the diagram.

21.


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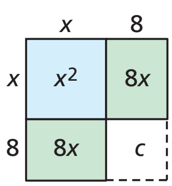

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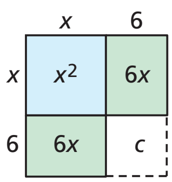

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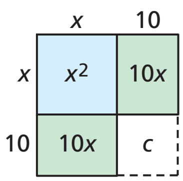


In Exercises 25–36, solve the equation by completing the square. *(See Examples 3 and 4.)*

25. $x^2 + 6x + 3 = 0$
26. $s^2 + 2s - 6 = 0$
27. $x^2 + 4x - 2 = 0$
28. $t^2 - 8t - 5 = 0$
29. $z(z + 9) = 1$
30. $x(x + 8) = -20$
31. $7t^2 + 28t + 56 = 0$
32. $6r^2 + 6r + 12 = 0$
33. $5x(x + 6) = -50$
34. $4w(w - 3) = 24$
35. $4x^2 - 30x = 12 + 10x$
36. $3s^2 + 8s = 2s - 9$

37. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

$$4x^2 + 24x - 11 = 0$$
$$4(x^2 + 6x) = 11$$
$$4(x^2 + 6x + 9) = 11 + 9$$
$$4(x + 3)^2 = 20$$
$$(x + 3)^2 = 5$$
$$x + 3 = \pm\sqrt{5}$$
$$x = -3 \pm \sqrt{5}$$

38. **ERROR ANALYSIS** Describe and correct the error in finding the value of c that makes the expression a perfect square trinomial.

$$x^2 + 30x + c$$
$$x^2 + 30x + \frac{30}{2}$$
$$x^2 + 30x + 15$$

39. **WRITING** Can you solve an equation by completing the square when the equation has two imaginary solutions? Explain.

40. **ABSTRACT REASONING** Which of the following are solutions of the equation $x^2 - 2ax + a^2 = b^2$? Justify your answers.

Ⓐ ab Ⓑ $-a - b$

Ⓒ b Ⓓ a

Ⓔ $a - b$ Ⓕ $a + b$

USING STRUCTURE In Exercises 41–50, determine whether you would use factoring, square roots, or completing the square to solve the equation. Explain your reasoning. Then solve the equation.

41. $x^2 - 4x - 21 = 0$

42. $x^2 + 13x + 22 = 0$

43. $(x + 4)^2 = 16$

44. $(x - 7)^2 = 9$

45. $x^2 + 12x + 36 = 0$

46. $x^2 - 16x + 64 = 0$

47. $2x^2 + 4x - 3 = 0$

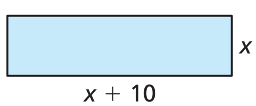
48. $3x^2 + 12x + 1 = 0$

49. $x^2 - 100 = 0$

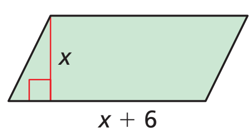
50. $4x^2 - 20 = 0$

MATHEMATICAL CONNECTIONS In Exercises 51–54, find the value of *x*.

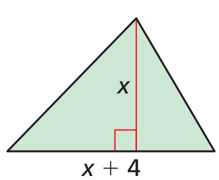
51. Area of rectangle = 50



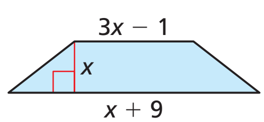
52. Area of parallelogram = 48



53. Area of triangle = 40



54. Area of trapezoid = 20



In Exercises 55–62, write the quadratic function in vertex form. Then identify the vertex. *(See Example 5.)*

55. $f(x) = x^2 - 8x + 19$

56. $g(x) = x^2 - 4x - 1$

57. $g(x) = x^2 + 12x + 37$

58. $h(x) = x^2 + 20x + 90$

59. $h(x) = x^2 + 2x - 48$

60. $f(x) = x^2 + 6x - 16$

61. $f(x) = x^2 - 3x + 4$

62. $g(x) = x^2 + 7x + 2$

63. **MODELING WITH MATHEMATICS** While marching, a drum major tosses a baton into the air and catches it. The height h (in feet) of the baton t seconds after it is thrown can be modeled by the function $h = -16t^2 + 32t + 6$. *(See Example 6.)*

    a. Find the maximum height of the baton.

    b. The drum major catches the baton when it is 4 feet above the ground. How long is the baton in the air?

64. **MODELING WITH MATHEMATICS** A firework explodes when it reaches its maximum height. The height h (in feet) of the firework t seconds after it is launched can be modeled by $h = -\frac{500}{9}t^2 + \frac{1000}{3}t + 10$. What is the maximum height of the firework? How long is the firework in the air before it explodes?

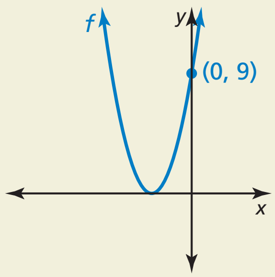
65. **COMPARING METHODS** A skateboard shop sells about 50 skateboards per week when the advertised price is charged. For each \$1 decrease in price, one additional skateboard per week is sold. The shop's revenue can be modeled by $y = (70 - x)(50 + x)$.

    a. Use the intercept form of the function to find the maximum weekly revenue.

    b. Write the function in vertex form to find the maximum weekly revenue.

    c. Which way do you prefer? Explain your reasoning.

66. HOW DO YOU SEE IT? The graph of the function $f(x) = (x - h)^2$ is shown. What is the x-intercept? Explain your reasoning.

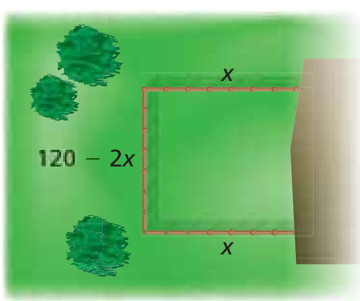


67. WRITING At Buckingham Fountain in Chicago, the height h (in feet) of the water above the main nozzle can be modeled by $h = -16t^2 + 89.6t$, where t is the time (in seconds) since the water has left the nozzle. Describe three different ways you could find the maximum height the water reaches. Then choose a method and find the maximum height of the water.

68. PROBLEM SOLVING A farmer is building a rectangular pen along the side of a barn for animals. The barn will serve as one side of the pen. The farmer has 120 feet of fence to enclose an area of 1512 square feet and wants each side of the pen to be at least 20 feet long.

a. Write an equation that represents the area of the pen.

b. Solve the equation in part (a) to find the dimensions of the pen.



69. MAKING AN ARGUMENT Your friend says the equation $x^2 + 10x = -20$ can be solved by either completing the square or factoring. Is your friend correct? Explain.

70. THOUGHT PROVOKING Write a function g in standard form whose graph has the same x-intercepts as the graph of $f(x) = 2x^2 + 8x + 2$. Find the zeros of each function by completing the square. Graph each function.

71. CRITICAL THINKING Solve $x^2 + bx + c = 0$ by completing the square. Your answer will be an expression for x in terms of b and c.

72. DRAWING CONCLUSIONS In this exercise, you will investigate the graphical effect of completing the square.

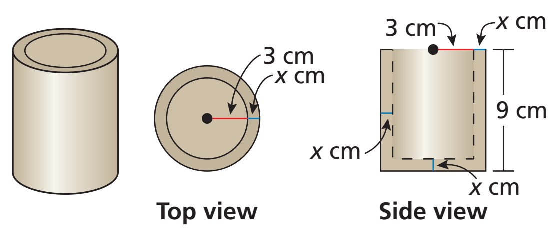
a. Graph each pair of functions in the same coordinate plane.

$y = x^2 + 2x$ $\quad$ $y = x^2 - 6x$

$y = (x + 1)^2$ $\quad$ $y = (x - 3)^2$

b. Compare the graphs of $y = x^2 + bx$ and $y = \left(x + \frac{b}{2}\right)^2$. Describe what happens to the graph of $y = x^2 + bx$ when you complete the square.

73. MODELING WITH MATHEMATICS In your pottery class, you are given a lump of clay with a volume of 200 cubic centimeters and are asked to make a cylindrical pencil holder. The pencil holder should be 9 centimeters high and have an inner radius of 3 centimeters. What thickness x should your pencil holder have if you want to use all of the clay?



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality. Graph the solution. *(Skills Review Handbook)*

74. $2x - 3 < 5$ **75.** $4 - 8y \geq 12$ **76.** $\frac{n}{3} + 6 > 1$ **77.** $-\frac{2s}{5} \leq 8$

Graph the function. Label the vertex, axis of symmetry, and x-intercepts. *(Section 2.2)*

78. $g(x) = 6(x - 4)^2$ **79.** $h(x) = 2x(x - 3)$

80. $f(x) = x^2 + 2x + 5$ **81.** $f(x) = 2(x + 10)(x - 12)$