6.3 Medians and Altitudes of Triangles

COMMON CORE

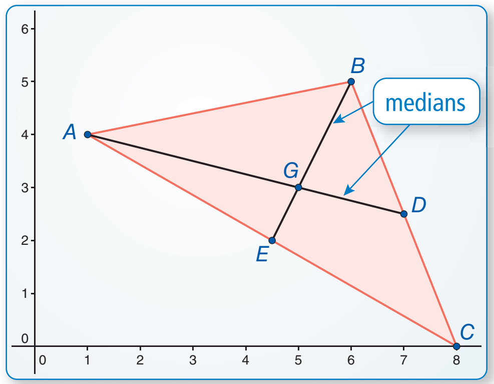
Learning Standard
HSG-CO.C.10

Essential Question What conjectures can you make about the medians and altitudes of a triangle?

EXPLORATION 1 Finding Properties of the Medians of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Plot the midpoint of $\overline{BC}$ and label it D. Draw $\overline{AD}$, which is a *median* of $\triangle ABC$. Construct the medians to the other two sides of $\triangle ABC$.



Sample
Points
$A(1, 4)$
$B(6, 5)$
$C(8, 0)$
$D(7, 2.5)$
$E(4.5, 2)$
$G(5, 3)$

b. What do you notice about the medians? Drag the vertices to change $\triangle ABC$. Use your observations to write a conjecture about the medians of a triangle.

c. In the figure above, point G divides each median into a shorter segment and a longer segment. Find the ratio of the length of each longer segment to the length of the whole median. Is this ratio always the same? Justify your answer.

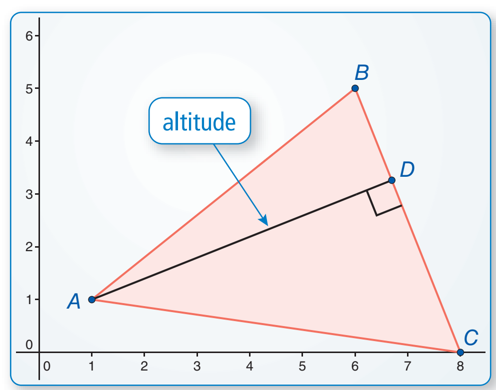
EXPLORATION 2 Finding Properties of the Altitudes of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the perpendicular segment from vertex A to $\overline{BC}$. Label the endpoint D. $\overline{AD}$ is an *altitude* of $\triangle ABC$.

b. Construct the altitudes to the other two sides of $\triangle ABC$. What do you notice?

c. Write a conjecture about the altitudes of a triangle. Test your conjecture by dragging the vertices to change $\triangle ABC$.



LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

3. What conjectures can you make about the medians and altitudes of a triangle?

4. The length of median $\overline{RU}$ in $\triangle RST$ is 3 inches. The point of concurrency of the three medians of $\triangle RST$ divides $\overline{RU}$ into two segments. What are the lengths of these two segments?

6.3 Lesson

Core Vocabulary

median of a triangle, *p. 320*
centroid, *p. 320*
altitude of a triangle, *p. 321*
orthocenter, *p. 321*

Previous
midpoint
concurrent
point of concurrency

What You Will Learn

- Use medians and find the centroids of triangles.
- Use altitudes and find the orthocenters of triangles.

Using the Median of a Triangle

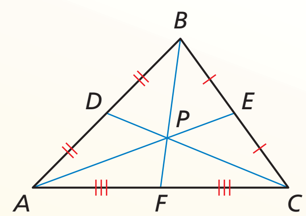
A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the **centroid**, is inside the triangle.

Theorem

Theorem 6.7 Centroid Theorem

The centroid of a triangle is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of $\triangle ABC$ meet at point P, and $AP = \frac{2}{3}AE$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CD$.



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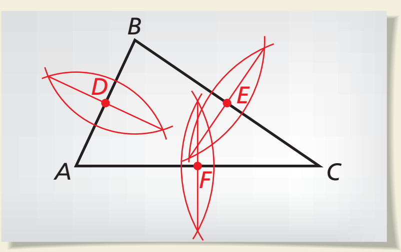
CONSTRUCTION Finding the Centroid of a Triangle

Use a compass and straightedge to construct the medians of $\triangle ABC$.

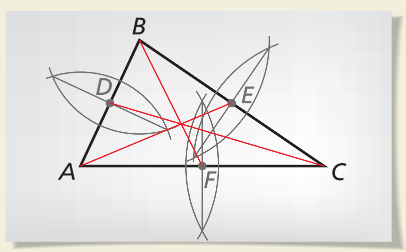
SOLUTION

Step 1



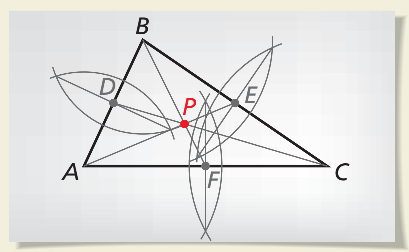
Find midpoints Draw $\triangle ABC$. Find the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Label the midpoints of the sides D, E, and F, respectively.

Step 2



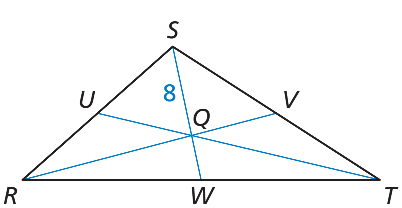
Draw medians Draw $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. These are the three medians of $\triangle ABC$.

Step 3



Label a point Label the point where $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ intersect as P. This is the centroid.

EXAMPLE 1 Using the Centroid of a Triangle



In $\triangle RST$, point Q is the centroid, and $SQ = 8$. Find QW and SW.

SOLUTION

$SQ = \frac{2}{3}SW$	Centroid Theorem
$8 = \frac{2}{3}SW$	Substitute 8 for SQ.
$12 = SW$	Multiply each side by the reciprocal, $\frac{3}{2}$.

Then $QW = SW - SQ = 12 - 8 = 4$.

- So, $QW = 4$ and $SW = 12$.

EXAMPLE 2 Finding the Centroid of a Triangle

Find the coordinates of the centroid of $\triangle RST$ with vertices $R(2, 1)$, $S(5, 8)$, and $T(8, 3)$.

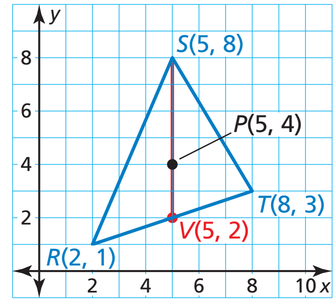
SOLUTION

Step 1 Graph $\triangle RST$.

Step 2 Use the Midpoint Formula to find the midpoint V of $\overline{RT}$ and sketch median $\overline{SV}$.

$$V\left(\frac{2+8}{2}, \frac{1+3}{2}\right) = (5, 2)$$

Step 3 Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.



The distance from vertex $S(5, 8)$ to $V(5, 2)$ is $8 - 2 = 6$ units.
So, the centroid is $\frac{2}{3}(6) = 4$ units down from vertex S on $\overline{SV}$.

▶ So, the coordinates of the centroid P are $(5, 8 - 4)$, or $(5, 4)$.

FINDING AN ENTRY POINT

The median $\overline{SV}$ is chosen in Example 2 because it is easier to find a distance on a vertical segment.

JUSTIFYING CONCLUSIONS

You can check your result by using a different median to find the centroid.

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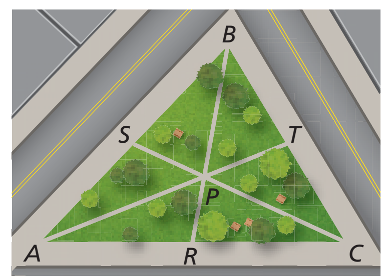
There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point *P*.



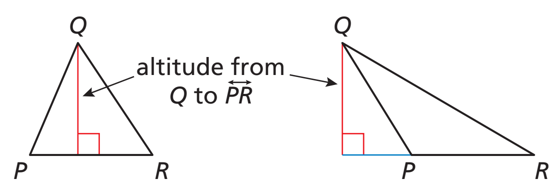
1. Find PS and PC when $SC = 2100$ feet.
2. Find TC and BC when $BT = 1000$ feet.
3. Find PA and TA when $PT = 800$ feet.

Find the coordinates of the centroid of the triangle with the given vertices.

4. $F(2, 5)$, $G(4, 9)$, $H(6, 1)$
5. $X(-3, 3)$, $Y(1, 5)$, $Z(-1, -2)$

Using the Altitude of a Triangle

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.



READING

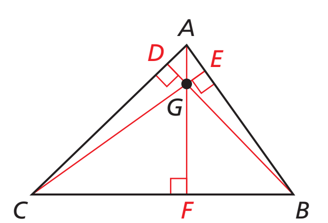
In the area formula for a triangle, $A = \frac{1}{2}bh$, you can use the length of any side for the base b. The height h is the length of the altitude to that side from the opposite vertex.

Core Concept

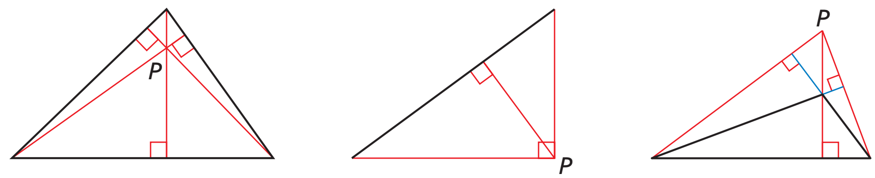
Orthocenter

The lines containing the altitudes of a triangle are concurrent. This point of concurrency is the **orthocenter** of the triangle.

The lines containing $\overline{AF}$, $\overline{BD}$, and $\overline{CE}$ meet at the orthocenter G of $\triangle ABC$.



As shown below, the location of the orthocenter P of a triangle depends on the type of triangle.



Acute triangle
P is inside triangle.

Right triangle
P is on triangle.

Obtuse triangle
P is outside triangle.

READING

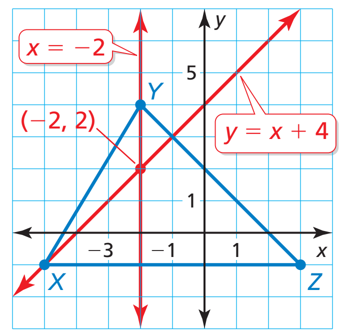
The altitudes are shown in red. Notice that in the right triangle, the legs are also altitudes. The altitudes of the obtuse triangle are extended to find the orthocenter.

EXAMPLE 3 Finding the Orthocenter of a Triangle

Find the coordinates of the orthocenter of $\triangle XYZ$ with vertices $X(-5, -1)$, $Y(-2, 4)$, and $Z(3, -1)$.

SOLUTION

Step 1 Graph $\triangle XYZ$.



Step 2 Find an equation of the line that contains the altitude from Y to $\overline{XZ}$. Because $\overline{XZ}$ is horizontal, the altitude is vertical. The line that contains the altitude passes through $Y(-2, 4)$. So, the equation of the line is $x = -2$.

Step 3 Find an equation of the line that contains the altitude from X to $\overline{YZ}$.

$$\text{slope of } \overleftrightarrow{YZ} = \frac{-1 - 4}{3 - (-2)} = -1$$

Because the product of the slopes of two perpendicular lines is -1, the slope of a line perpendicular to $\overleftrightarrow{YZ}$ is 1. The line passes through $X(-5, -1)$.

$y = mx + b$ Use slope-intercept form.

$-1 = 1(-5) + b$ Substitute -1 for y, 1 for m, and -5 for x.

$4 = b$ Solve for b.

So, the equation of the line is $y = x + 4$.

Step 4 Find the point of intersection of the graphs of the equations $x = -2$ and $y = x + 4$.

Substitute -2 for x in the equation $y = x + 4$. Then solve for y.

$y = x + 4$ Write equation.

$y = -2 + 4$ Substitute -2 for x.

$y = 2$ Solve for y.

▶ So, the coordinates of the orthocenter are $(-2, 2)$.

Monitoring Progress

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Tell whether the orthocenter of the triangle with the given vertices is *inside*, *on*, or *outside* the triangle. Then find the coordinates of the orthocenter.

6. $A(0, 3)$, $B(0, -2)$, $C(6, -3)$

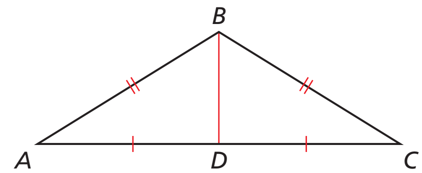
7. $J(-3, -4)$, $K(-3, 4)$, $L(5, 4)$

In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for any vertex.

EXAMPLE 4 Proving a Property of Isosceles Triangles

Prove that the median from the vertex angle to the base of an isosceles triangle is an altitude.

SOLUTION



Given $\triangle ABC$ is isosceles, with base $\overline{AC}$.
$\overline{BD}$ is the median to base $\overline{AC}$.

Prove $\overline{BD}$ is an altitude of $\triangle ABC$.

Paragraph Proof Legs $\overline{AB}$ and $\overline{BC}$ of isosceles $\triangle ABC$ are congruent. $\overline{CD} \cong \overline{AD}$ because $\overline{BD}$ is the median to $\overline{AC}$. Also, $\overline{BD} \cong \overline{BD}$ by the Reflexive Property of Congruence (Thm. 2.1). So, $\triangle ABD \cong \triangle CBD$ by the SSS Congruence Theorem (Thm. 5.8). $\angle ADB \cong \angle CDB$ because corresponding parts of congruent triangles are congruent. Also, $\angle ADB$ and $\angle CDB$ are a linear pair. $\overline{BD}$ and $\overline{AC}$ intersect to form a linear pair of congruent angles, so $\overline{BD} \perp \overline{AC}$ and $\overline{BD}$ is an altitude of $\triangle ABC$.

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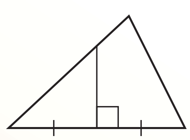
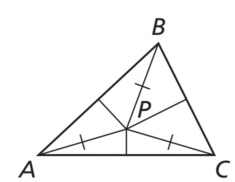
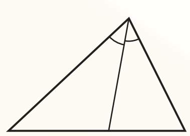
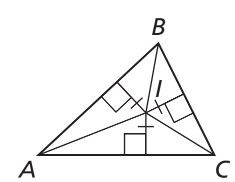
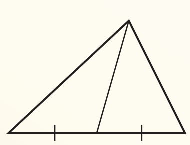
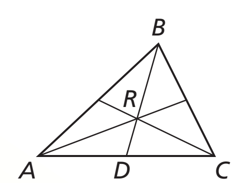
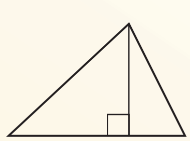
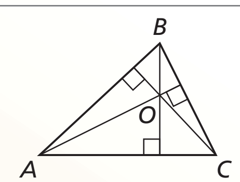
8. WHAT IF? In Example 4, you want to show that median $\overline{BD}$ is also an angle bisector. How would your proof be different?

Concept Summary

Segments, Lines, Rays, and Points in Triangles

	Example	Point of Concurrency	Property	Example
perpendicular bisector		circumcenter	The circumcenter P of a triangle is equidistant from the vertices of the triangle.	
angle bisector		incenter	The incenter I of a triangle is equidistant from the sides of the triangle.	
median		centroid	The centroid R of a triangle is two thirds of the distance from each vertex to the midpoint of the opposite side.	
altitude		orthocenter	The lines containing the altitudes of a triangle are concurrent at the orthocenter O.	

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

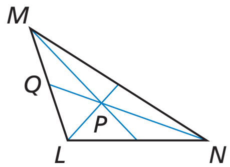
1. **VOCABULARY** Name the four types of points of concurrency. Which lines intersect to form each of the points?

2. **COMPLETE THE SENTENCE** The length of a segment from a vertex to the centroid is ____________ the length of the median from that vertex.

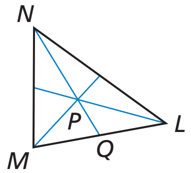
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, point P is the centroid of $\triangle LMN$. Find PN and QP. *(See Example 1.)*

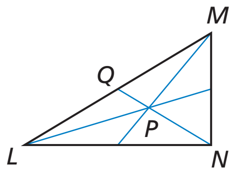
3. $QN = 9$



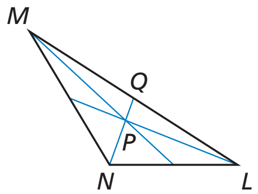
4. $QN = 21$



5. $QN = 30$

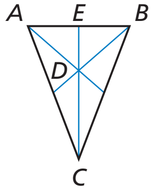


6. $QN = 42$

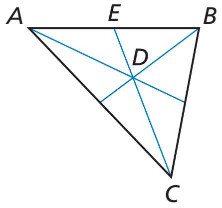


In Exercises 7–10, point D is the centroid of $\triangle ABC$. Find CD and CE.

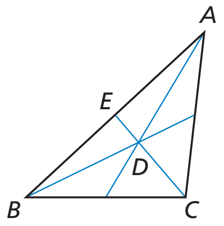
7. $DE = 5$



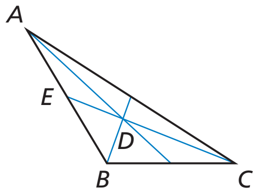
8. $DE = 11$



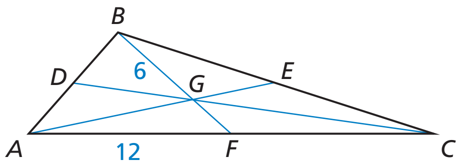
9. $DE = 9$



10. $DE = 15$



In Exercises 11–14, point G is the centroid of $\triangle ABC$. $BG = 6$, $AF = 12$, and $AE = 15$. Find the length of the segment.



11. $\overline{FC}$

12. $\overline{BF}$

13. $\overline{AG}$

14. $\overline{GE}$

In Exercises 15–18, find the coordinates of the centroid of the triangle with the given vertices. *(See Example 2.)*

15. $A(2, 3)$, $B(8, 1)$, $C(5, 7)$

16. $F(1, 5)$, $G(-2, 7)$, $H(-6, 3)$

17. $S(5, 5)$, $T(11, -3)$, $U(-1, 1)$

18. $X(1, 4)$, $Y(7, 2)$, $Z(2, 3)$

In Exercises 19–22, tell whether the orthocenter is *inside*, *on*, or *outside* the triangle. Then find the coordinates of the orthocenter. *(See Example 3.)*

19. $L(0, 5)$, $M(3, 1)$, $N(8, 1)$

20. $X(-3, 2)$, $Y(5, 2)$, $Z(-3, 6)$

21. $A(-4, 0)$, $B(1, 0)$, $C(-1, 3)$

22. $T(-2, 1)$, $U(2, 1)$, $V(0, 4)$

CONSTRUCTION In Exercises 23–26, draw the indicated triangle and find its centroid and orthocenter.

23. isosceles right triangle

24. obtuse scalene triangle

25. right scalene triangle

26. acute isosceles triangle

ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in finding DE. Point D is the centroid of $\triangle ABC$.**

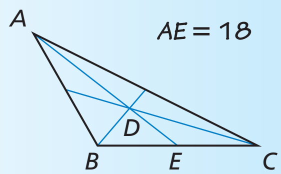
27.

$DE = \frac{2}{3}AE$

$DE = \frac{2}{3}(18)$

$DE = 12$

$AE = 18$



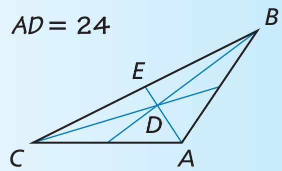
28.

$DE = \frac{2}{3}AD$

$DE = \frac{2}{3}(24)$

$DE = 16$

$AD = 24$



PROOF **In Exercises 29 and 30, write a proof of the statement.** *(See Example 4.)*

29. The angle bisector from the vertex angle to the base of an isosceles triangle is also a median.

30. The altitude from the vertex angle to the base of an isosceles triangle is also a perpendicular bisector.

CRITICAL THINKING **In Exercises 31–36, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

31. The centroid is __________ on the triangle.

32. The orthocenter is __________ outside the triangle.

33. A median is __________ the same line segment as a perpendicular bisector.

34. An altitude is __________ the same line segment as an angle bisector.

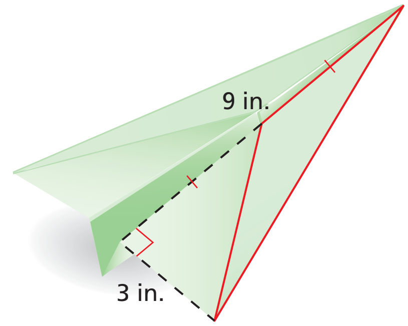
35. The centroid and orthocenter are __________ the same point.

36. The centroid is __________ formed by the intersection of the three medians.

37. **WRITING** Compare an altitude of a triangle with a perpendicular bisector of a triangle.

38. **WRITING** Compare a median, an altitude, and an angle bisector of a triangle.

39. **MODELING WITH MATHEMATICS** Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?

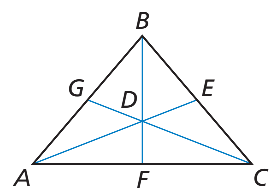


40. **ANALYZING RELATIONSHIPS** Copy and complete the statement for $\triangle DEF$ with centroid K and medians $\overline{DH}$, $\overline{EJ}$, and $\overline{FG}$.

a. $EJ =$ _____ KJ **b.** $DK =$ _____ KH

c. $FG =$ _____ KF **d.** $KG =$ _____ FG

MATHEMATICAL CONNECTIONS **In Exercises 41–44, point D is the centroid of $\triangle ABC$. Use the given information to find the value of x.**



41. $BD = 4x + 5$ and $BF = 9x$

42. $GD = 2x - 8$ and $GC = 3x + 3$

43. $AD = 5x$ and $DE = 3x - 2$

44. $DF = 4x - 1$ and $BD = 6x + 4$

45. **MATHEMATICAL CONNECTIONS** Graph the lines on the same coordinate plane. Find the centroid of the triangle formed by their intersections.

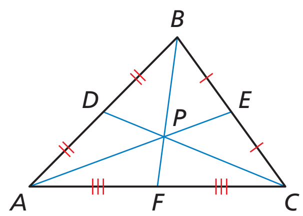
$y_1 = 3x - 4$

$y_2 = \frac{3}{4}x + 5$

$y_3 = -\frac{3}{2}x - 4$

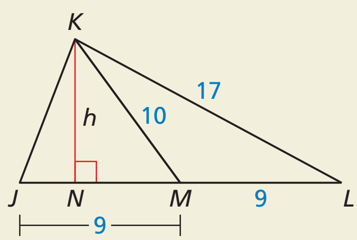
46. **CRITICAL THINKING** In what type(s) of triangles can a vertex be one of the points of concurrency of the triangle? Explain your reasoning.

47. **WRITING EQUATIONS** Use the numbers and symbols to write three different equations for PE.



PE	AE	AP	$+$	$-$
$=$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

48. **HOW DO YOU SEE IT?** Use the figure.



a. What type of segment is $\overline{KM}$? Which point of concurrency lies on $\overline{KM}$?

b. What type of segment is $\overline{KN}$? Which point of concurrency lies on $\overline{KN}$?

c. Compare the areas of $\triangle JKM$ and $\triangle KLM$. Do you think the areas of the triangles formed by the median of any triangle will always compare this way? Explain your reasoning.

49. **MAKING AN ARGUMENT** Your friend claims that it is possible for the circumcenter, incenter, centroid, and orthocenter to all be the same point. Do you agree? Explain your reasoning.

50. **DRAWING CONCLUSIONS** The center of gravity of a triangle, the point where a triangle can balance on the tip of a pencil, is one of the four points of concurrency. Draw and cut out a large scalene triangle on a piece of cardboard. Which of the four points of concurrency is the center of gravity? Explain.

51. **PROOF** Prove that a median of an equilateral triangle is also an angle bisector, perpendicular bisector, and altitude.

52. **THOUGHT PROVOKING** Construct an acute scalene triangle. Find the orthocenter, centroid, and circumcenter. What can you conclude about the three points of concurrency?

53. **CONSTRUCTION** Follow the steps to construct a nine-point circle. Why is it called a nine-point circle?

Step 1 Construct a large acute scalene triangle.

Step 2 Find the orthocenter and circumcenter of the triangle.

Step 3 Find the midpoint between the orthocenter and circumcenter.

Step 4 Find the midpoint between each vertex and the orthocenter.

Step 5 Construct a circle. Use the midpoint in Step 3 as the center of the circle, and the distance from the center to the midpoint of a side of the triangle as the radius.

54. **PROOF** Prove the statements in parts (a)−(c).

Given $\overline{LP}$ and $\overline{MQ}$ are medians of scalene $\triangle LMN$. Point R is on $\overrightarrow{LP}$ such that $\overline{LP} \cong \overline{PR}$. Point S is on $\overrightarrow{MQ}$ such that $\overline{MQ} \cong \overline{QS}$.

Prove **a.** $\overline{NS} \cong \overline{NR}$

b. $\overline{NS}$ and $\overline{NR}$ are both parallel to $\overline{LM}$.

c. R, N, and S are collinear.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether $\overline{AB}$ is parallel to $\overline{CD}$. *(Section 3.5)*

55. $A(5, 6)$, $B(-1, 3)$, $C(-4, 9)$, $D(-16, 3)$

56. $A(-3, 6)$, $B(5, 4)$, $C(-14, -10)$, $D(-2, -7)$

57. $A(6, -3)$, $B(5, 2)$, $C(-4, -4)$, $D(-5, 2)$

58. $A(-5, 6)$, $B(-7, 2)$, $C(7, 1)$, $D(4, -5)$